GM2-doc-2343

Effects of Scattering on Courant-Snyder Parameters

M.J. Syphers

November 13, 2014

Scattering of particles through the inflector at the entrance to the Muon g-2 Storage Ring has brought about questions regarding the matching of Courant-Snyder parameters from the beam line to those of the ring. It is known that the emittance of the beam will increase, but questions arise as to how much the emittance growth will be and what is the effect on the lattice parameters.

Imagine a beam with an initial phase space distribution (x, x') in one transverse degree of freedom characterized by Courant-Snyder parameters α_0, β_0, and γ_0 and an rms emittance, ϵ_0. In terms of the variances and covariances of the phase space variables of an assumed distribution with zero mean values, the rms emittance is given by

$$\epsilon_0 = \pi\sqrt{\langle x_0^2\rangle\langle {x_0'}^2\rangle - \langle x_0 x_0'\rangle^2}$$

where we purposely keep the "π" in the definition to (attempt to) avoid confusion. The Courant-Snyder parameters are then determined by:

$$\beta_0 = \frac{\pi\langle x_0^2\rangle}{\epsilon_0}, \qquad \gamma_0 = \frac{\pi\langle {x_0'}^2\rangle}{\epsilon_0}, \qquad \alpha_0 = -\frac{\pi\langle x_0 x_0'\rangle}{\epsilon_0}.$$

We assume that the distance through the material that is doing the scattering is thin enough such that the transverse position doesn't change significantly, but its angle changes due to multiple Coulomb interactions, say, by an amount $\Delta\theta$. For an individual particle, after scattering,

$$x' = x_0' + \Delta\theta$$

and hence,

$$\begin{aligned}
\langle x^2 \rangle &= \langle x_0^2 \rangle \\
\langle x'^2 \rangle &= \langle (x_0' + \Delta\theta)^2 \rangle \\
&= \langle {x_0'}^2 + \Delta\theta^2 + 2x_0'\Delta\theta \rangle \\
&= \langle {x_0'}^2 \rangle + \langle \Delta\theta^2 \rangle \\
\langle xx' \rangle &= \langle x_0(x_0' + \Delta\theta) \rangle = \langle x_0 x_0' \rangle
\end{aligned}$$

assuming that the scattering process is uncorrelated with the phase space variables. The new emittance after scattering is then given by,

$$(\epsilon/\pi)^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = \langle x_0^2 \rangle (\langle {x_0'}^2 \rangle + \theta_{rms}^2) - \langle x_0 x_0' \rangle^2$$

or,

$$\epsilon = \epsilon_0 \sqrt{1 + \frac{\langle x_0^2 \rangle \theta_{rms}^2}{(\epsilon_0/\pi)^2}}$$

where $\theta_{rms} = \langle \Delta\theta^2 \rangle^{1/2}$. Since $\beta_0 = \pi \langle x_0^2 \rangle / \epsilon_0$, then we can write

$$\epsilon = \epsilon_0 \sqrt{1 + \theta_{rms}^2 \frac{\beta_0}{(\epsilon_0/\pi)}}.$$

And, using this new emittance, we can describe the resulting distribution after the scattering by new Courant-Snyder parameters

$$\begin{aligned}
\beta &= \frac{\pi \langle x^2 \rangle}{\epsilon} = \beta_0 \ \frac{\epsilon_0}{\epsilon} \\
\alpha &= \alpha_0 \ \frac{\epsilon_0}{\epsilon} \\
\gamma &= \left(\gamma_0 + \frac{\theta_{rms}^2}{(\epsilon_0/\pi)} \right) \ \frac{\epsilon_0}{\epsilon}
\end{aligned}$$

One can check that indeed $\beta\gamma - \alpha^2 = \beta_0\gamma_0 - \alpha_0^2 = 1$.

Figure 1 shows an example of the phase space distribution before and after scattering where arbitrary initial parameters are $\alpha_0 = -2$, $\beta_0 = 20$ m, and the initial rms beam size is chosen to be 1 mm, for which the rms emittance $\epsilon = 0.05\ \pi$ mm-mr. The rms scattering angle is 0.25 mr in the example. The ellipses indicated by dotted lines correspond to 95% emittances ($= 6 \times \epsilon$ given above) for easier visibility. Here, the emittance growth is a factor of 5, and the amplitude functions β and α decrease by factors of 5, consistent with $\sqrt{1 + \pi\beta_0\theta_{rms}^2/\epsilon_0}$ for the parameters used.

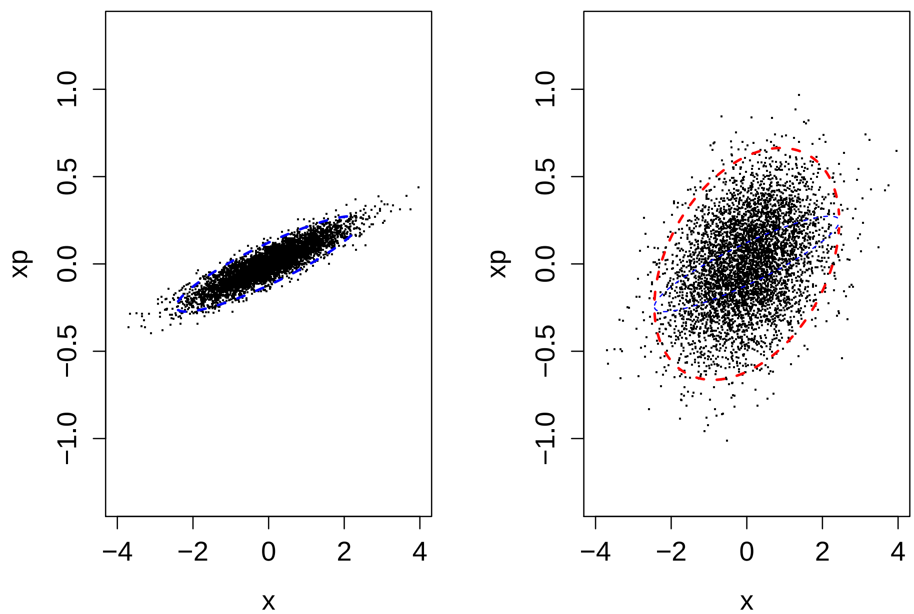

Figure 1: Phase space before/after (left/right) scattering. Blue/red lines indicate 95% emittances before/after scattering.

As a numerical example, perhaps pertinent to injection through the inflector for instance, consider the Coulomb scattering of the muon beam through material of thickness ℓ and with radiation length L_{rad}. The rms scattering angle is given approximately by

$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{\beta pc} \sqrt{\frac{\ell}{L_{rad}}}$$

Let's use $L_{rad} = 16$ g/cm^2 / (8 g/cm^3) = 2 cm for elements in the range of Cu, Sn, and Ti, and use $\ell = 2$ cm. Then, for a 3.09 GeV muon, the scattering angle is on the scale of 14/3.1 mr $\approx$ 4 mr. Focusing an $\epsilon =$ 40 π mm-mr (95%, say) beam to a 4 mm (rms) spot size gives for the Courant-Snyder parameters: $\beta = 4^2/(40/6)$ m $= 2.4$ m, $\alpha = 0$ (a waist), and $\gamma = 0.4$/m. Upon scattering, the emittance will grow by a factor of $\epsilon/\epsilon_0 = [1+4^2 \times 2.4/(40/6)]^{1/2} = 2.6$. The amplitude function β will decrease by the same factor.

One could imagine building in the computation of such changes in emittance and beam line parameters upon passing through material when performing the usual optics calculations of the g-2 beam lines, for quick estimations.